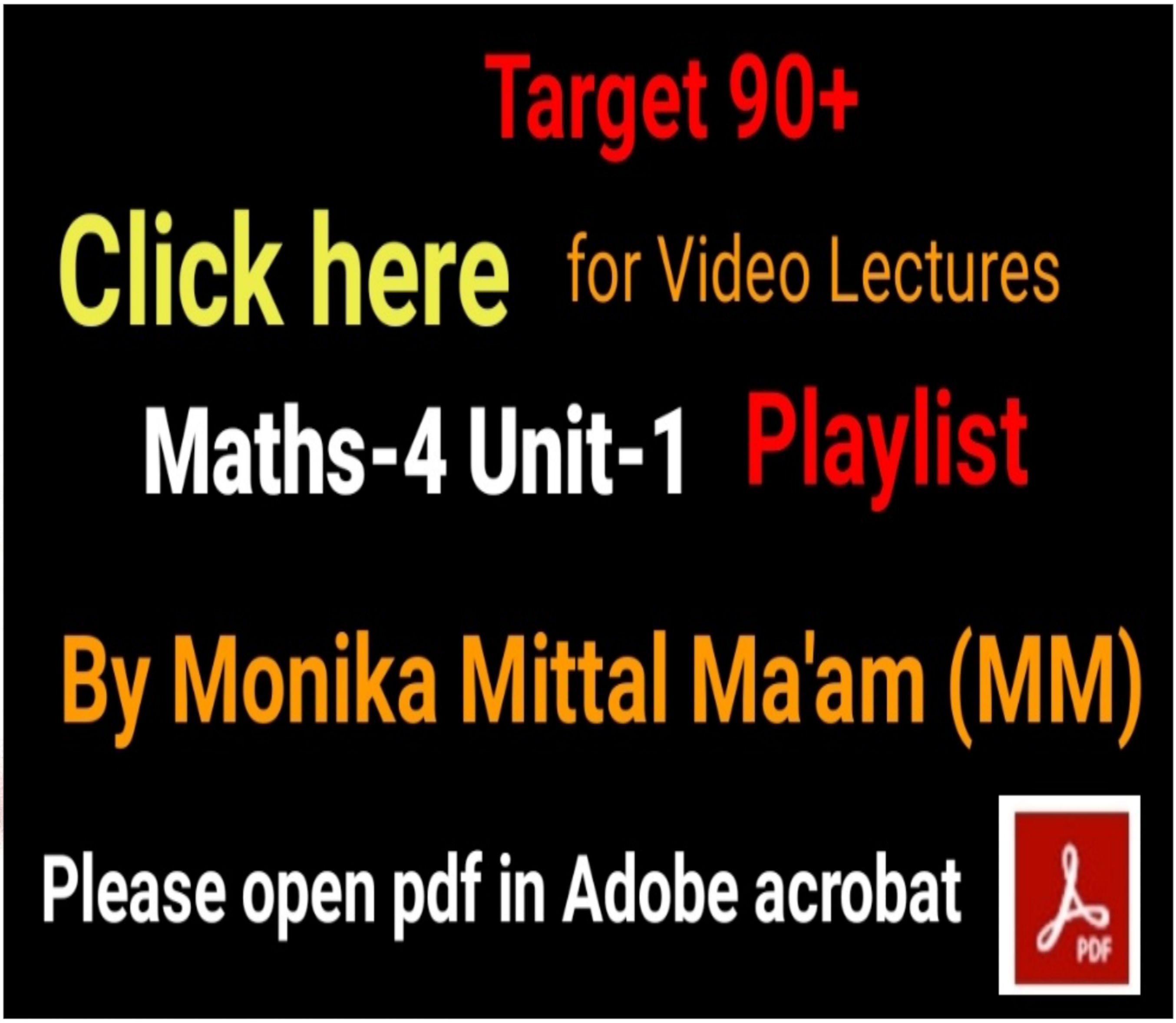
Maths-4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By Monika Mittal Ma'am (MM)
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Partial Differential Equation :-

An Equation containing partial derivatives of a function of two or more Variables (independent Variables) is called Partial Differential Equation.

Example :- $\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = xyz$ — (1)

$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$ — (2) $u \to$ dependent fn. $x, y \to$ indep. Var.

$y\frac{\partial z}{\partial x} + x\frac{\partial z}{\partial y} = 0$ — (3) $z \to$ dep, $x, y \to$ indep

$\left(\frac{\partial z}{\partial x}\right)^2 + \frac{\partial^3 z}{\partial y^3} = 2x\left(\frac{\partial z}{\partial x}\right)$ — (4)

Order :- Order of P.D.E. is order of highest partial derivative in equation.

ex- (1) Order → 1
ex -(2) Order → 2
ex- 3 Order → 1
ex - 4 Order → 3

Degree :- Degree of PDE is degree of highest order derivative, provided equation is free from fractional powers and radicals.

(1) $\frac{\partial z}{\partial x} + \frac{\partial z}{\partial y} = z$ $O = 1$, $D = 1$

(2) $\frac{\partial^2 u}{\partial x^2} = \left(\frac{\partial u}{\partial y} + 5u\right)^{\frac{1}{2}}$ $O = 2$ $D = 2$

$\left(\frac{\partial^2 u}{\partial x^2}\right)^2 = \left(\frac{\partial u}{\partial y} + 5u\right)$

Linear PDE :- A PDE is said to be linear, if dependent Variable and its partial derivative occur only in first degree and not multiplied.
Which is not linear is called Non-linear PDE

Example :-

(1) $\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} = xy \rightarrow$ Linear

(2) $u\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} = x \rightarrow$ Non-linear

(3) $\frac{\partial z}{\partial x}\frac{\partial z}{\partial y} = xy \rightarrow$ Non linear.

(4) $\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right) = 1 \rightarrow$ Non-Linear

Semi Linear PDE A first order PDE $f(x, y, z, p, q)=0$ is known as semi-linear, if it is in the form

$$P(x,y)\,p + Q(x,y)\,q = R(x,y,z)$$

$p = \frac{\partial z}{\partial x}$

$q = \frac{\partial z}{\partial y}$

where $p = \frac{\partial z}{\partial x}$, $q = \frac{\partial z}{\partial y}$

Example :- ① $xyp + xy^2q = x^2y^2z^2 \Rightarrow xy\frac{\partial z}{\partial x} + xy^2\frac{\partial z}{\partial y} = x^2y^2z^2$

(2) $xy\,p + yx^2\,q = xz^2$

Quasi Linear :- A first order PDE $f(x,y,z,p,q)=0$ is known as Quasi-linear equation., if the given eqⁿ is of form

$$P(x,y,z)\,p + Q(x,y,z)\,q = R(x,y,z)$$

ex. $x^2z\,p + yz\,q = xy\sin z$ ✓

$(x^2-yz)p + (y^2-zx)q = z^2-xy$ ✓

L-2 Engg. Maths-4 (Unit-1)

Origin of Partial Differential Equation

⟹ Formation of PDE by eliminating arbitrary constant

⟹ Formation of PDE by eliminating arbitrary functions.

Rule-1 - By elimination of Arbitrary Constants.

let $f(x, y, z, a, b) = 0$ —(1) , a, b → arbitrary const

z → dep. Var

x, y → Indep. Var

$\frac{\partial z}{\partial x} = p$

$\frac{\partial z}{\partial y} = q$

Diff eqⁿ (1) w.r.t 'x', partially

$$\frac{\partial f}{\partial x} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial x} = 0 \Rightarrow \frac{\partial f}{\partial x} + \frac{\partial f}{\partial z}p = 0 \quad \text{—(2)}$$

Diff eqⁿ (1) w.r.t 'y' partially

$$\frac{\partial f}{\partial y} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial y} = 0 \Rightarrow \frac{\partial f}{\partial y} + \frac{\partial f}{\partial z}q = 0 \quad \text{—(3)}$$

eliminate a, b from (1), (2), (3)

$$F(x, y, z, p, q) = 0$$

Case-I When no. of arbitrary constants is less than No. of Independent Variables, then elimination gives rise to More than One PDE of Order One

No of arbitrary const < No of Independent Var. ⟹ More than One PDE of first Order

Ex. form. PDE, $z = ax + y$ —(1)

Diff w.r.t 'x', partially

$$\frac{\partial z}{\partial x} = a \quad \text{—(2)}$$

Diff eqⁿ (1) w.r.t 'y'

$$\frac{\partial z}{\partial y} = 1 \quad \text{—(3)} ✓$$

PDE

Eqⁿ (1) $z = \frac{\partial z}{\partial x}x + y \Rightarrow z = px + y$ —(4) ✓

Case-II :- When No. of arbitrary constants is equal to No. of independent Var., then elimination of arbitrary constants shall give rise to Unique PDE of first order

No of arb. const. = No of Indep. Var $\Rightarrow$ Unique PDE of first order

Ex. Form PDE, if $az+b = a^2x + y$ —— ① , | $a, b \to 2$; $x, y \to 2$

Diff eq^n ① w.r.t 'x' partially

$$a\frac{\partial z}{\partial x} = a^2 \Rightarrow \frac{\partial z}{\partial x} = a \quad \text{—— ②}$$

Diff eq^n ① w.r.t 'y' partially,

$$a\frac{\partial z}{\partial y} = 1 \Rightarrow \frac{\partial z}{\partial y} = \frac{1}{a} \quad \text{—(3)}$$

Multiply eq^n (2) & (3) $\frac{\partial z}{\partial x} \times \frac{\partial z}{\partial y} = a \times \frac{1}{a} = 1$

$$\boxed{pq = 1}$$

Ex. form PDE, if $z = (x^2+a)(y^2+b)$, where a, b are arbitrary constants. [G.B.T.U 2012]

| $a, b \to 2$; $x, y \to 2$

$$z = (x^2+a)(y^2+b) \quad \text{—— ①}$$

Diff eq^n (1) w.r.t 'x'

$$\frac{\partial z}{\partial x} = 2x(y^2+b) \Rightarrow (y^2+b) = \frac{p}{2x} \quad \text{—(2)}$$

Diff eq^n ① w.r.t 'y'

$$\frac{\partial z}{\partial y} = (x^2+a)\,2y \Rightarrow (x^2+a) = \frac{q}{2y} \quad \text{—③}$$

$$\text{Eq}^n \text{ ①} \Rightarrow z = \frac{q}{2y} \times \frac{p}{2x} \Rightarrow \boxed{4xyz = pq}$$

Ex. Form PDE, if $z = c_1 xy + c_2$, where c_1, c_2 are arbitrary constants, [MTU 2013]

$$z = c_1 xy + c_2 \quad \text{—— (1)}$$

Diff eqⁿ (1) w.r.t 'x' partially

$$\frac{\partial z}{\partial x} = c_1 y \quad \text{—— (2)} \quad \Rightarrow c_1 = \frac{p}{y}$$

Diff eqⁿ (1) w.r.t 'y' partially

$$\frac{\partial z}{\partial y} = c_1 x \quad \text{—— (3)} \quad \Rightarrow c_1 = \frac{q}{x}$$

$$\Rightarrow \frac{p}{y} = \frac{q}{x}$$

$$\Rightarrow xp = qy$$

$$\boxed{xp - qy = 0}$$

Ex. Form PDE, if $x^2 + y^2 + (z-a)^2 = b^2$, where a, b are arbitrary constants. [AKTU 2017]

$$x^2 + y^2 + (z-a)^2 = b^2 \quad \text{—— (1)}$$

Diff w.r.t 'x' eqⁿ (1)

$$\cancel{2}x + \cancel{2}(z-a)\frac{\partial z}{\partial x} = 0 \Rightarrow (z-a) = -\frac{x}{p} \quad \text{—— (2)}$$

Diff w.r.t 'y' eqⁿ (1)

$$\cancel{2}y + \cancel{2}(z-a)\frac{\partial z}{\partial y} = 0 \Rightarrow (z-a) = -\frac{y}{q} \quad \text{—— (3)}$$

$$\Rightarrow \frac{-x}{p} = \frac{-y}{q}$$

$$\Rightarrow \boxed{xq - py = 0}$$

Case-II When No of arbitrary constants > No of Independent Var $\Rightarrow$ PDE of order usually greater than one.

Notations :-

$$z = f(x,y) \Rightarrow \frac{\partial z}{\partial x} = p = z_x, \quad \frac{\partial z}{\partial y} = q = z_y$$

$$\frac{\partial^2 z}{\partial x^2} = r = z_{xx}, \quad \frac{\partial^2 z}{\partial x \partial y} = s = z_{xy}, \quad \frac{\partial^2 z}{\partial y^2} = t = z_{yy}$$

Rule-2 By Elimination of arbitrary functions.

(1) An Eq^n containing One arbitrary function ⇒ then First Order PDE.

(2) An Eq^n containing Two Arbitrary functions ⇒ then PDE of Higher Order greater than One

Ex.1 Form PDE, if $z = f(x^2 - y^2)$ [UPTU 2015]

Diff w.r.t 'x' partially

$$\frac{\partial z}{\partial x} = f'(x^2 - y^2) \cdot 2x \quad \text{--- (1)}$$

Diff w.r.t 'y' $\frac{\partial z}{\partial y} = f'(x^2-y^2)(-2y)$ --- (2)

Divide eq^n (1) by (2) $\frac{p}{q} = \frac{\cancel{f'(x^2-y^2)}2x}{\cancel{f'(x^2-y^2)}(-2y)} \Rightarrow \frac{p}{q} = \frac{x}{-y}$

$$\boxed{qx + yp = 0}$$

Ex.2 Form PDE, if $z = f(x+at) + g(x-at)$

No of arb. fn = 2

Diff eq^n w.r.t 'x'

$$\frac{\partial z}{\partial x} = f'(x+at).1 + g'(x-at).1 \quad \text{--- (1)}$$

$$\frac{\partial^2 z}{\partial x^2} = f''(x+at).1 + g''(x-at).1 \quad \text{--- (2)} \checkmark$$

Diff eq^n w.r.t 't'

$$\frac{\partial z}{\partial t} = f'(x+at)a + g'(x-at)(-a) \quad \text{--- (3)}$$

$$\frac{\partial^2 z}{\partial t^2} = f''(x+at)a^2 + g''(x-at)(-a)^2$$

$$\frac{\partial^2 z}{\partial t^2} = a^2[\underline{f''(x+at) + g''(x-at)}] \quad \text{--- (4)} \checkmark$$

$$\boxed{\frac{\partial^2 z}{\partial t^2} = a^2 \frac{\partial^2 z}{\partial x^2}}$$

L-3 (Unit-1)

Lagrange's Equation

A Quasi-linear PDE of order one is of form

$$P(x,y,z)\,p + Q(x,y,z)\,q = R(x,y,z) \qquad p=\frac{\partial z}{\partial x},\ q=\frac{\partial z}{\partial y}$$

where P, Q, R are functions of x, y, z.

Such a differential Equation is called Lagrange Eq^n.

$\Rightarrow$ The general solution is given by

$$\boxed{\phi(u,v)=0}$$

where ϕ is arbitrary function and

$u(x,y,z)=c_1$ and $v(x,y,z)=c_2$ are two linearly independent solution of

$$\frac{dx}{P}=\frac{dy}{Q}=\frac{dz}{R} \Rightarrow \text{Aux. Eq}^n.$$

Working Steps:-

S-1 $\Rightarrow$ Put given linear PDE of first order in standard form $Pp + Qq = R$

Form Lagrange's Aux. Eq^n $\boxed{\frac{dx}{P}=\frac{dy}{Q}=\frac{dz}{R}}$ Aux. Eq^n.

S-2 $\Rightarrow$ Solve and get two independent sol^n.

$u(x,y,z)=c_1$, $v(x,y,z)=c_2$.

S-3 $\Rightarrow$ The general solution can be written in forms $\phi(u,v)=0$ OR $u=f_1(v)$ OR $v=f_2(u)$

Solve, By Using

$\Rightarrow$ Grouping ✓

$\Rightarrow$ Multipliers ✓

$\Rightarrow$ OR Both

Type-I Solution by Using Grouping

$$\phi(u,v)=0$$ L-3-2

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$$
I II III

(I, II), (I, III)
(II, III)

Ex. 1 Solve $yzp + zxq = xy$

$Pp + Qq = R$

$P = yz$, $Q = zx$
$R = xy$

$u(x,y,z) = C_1$
$v(x,y,z) = C_2$

Aux. Eqn. $\Rightarrow \frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$

$$\Rightarrow \boxed{\frac{dx}{yz} = \frac{dy}{zx} = \frac{dz}{xy}}$$
I II III

(I, II) $\frac{dx}{y\cancel{z}} = \frac{dy}{\cancel{z}x} \Rightarrow x\,dx = y\,dy \Rightarrow \frac{x^2}{2} = \frac{y^2}{2} + C_1$

$\Rightarrow \boxed{x^2 - y^2 = C_1}$

(II, III) $\frac{dy}{z\cancel{x}} = \frac{dz}{\cancel{x}y} \Rightarrow y\,dy = z\,dz \Rightarrow \frac{y^2}{2} - \frac{z^2}{2} = C_2$

$\boxed{y^2 - z^2 = C_2} \Rightarrow \phi(x^2-y^2, y^2-z^2) = 0$

Ex. 2 Solve $p\sqrt{x} + q\sqrt{y} = \sqrt{z}$

$Pp + Qq = R$

$P = \sqrt{x}$, $Q = \sqrt{y}$
$R = \sqrt{z}$.

Aux eqn $\Rightarrow \frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$

$$\Rightarrow \frac{dx}{\sqrt{x}} = \frac{dy}{\sqrt{y}} = \frac{dz}{\sqrt{z}}$$
I II III

(I, II) $\Rightarrow \frac{dx}{\sqrt{x}} = \frac{dy}{\sqrt{y}} \Rightarrow 2\sqrt{x} = 2\sqrt{y} + C_1$

$\Rightarrow \boxed{\sqrt{x} - \sqrt{y} = C_1}$

(I, III) $\Rightarrow \frac{dx}{\sqrt{x}} = \frac{dz}{\sqrt{z}}$

$2\sqrt{x} = 2\sqrt{z} + C_2$

$\boxed{\sqrt{x} - \sqrt{z} = C_2}$

general soln
$\phi(u, v) = 0$

$\Rightarrow \phi(\sqrt{x} - \sqrt{y}, \sqrt{x} - \sqrt{z}) = 0$

Ex. solve $p+3q = 5z + \tan(y-3x)$ (AKTU 2017)

Aux eq^n $\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$ $Pp + Qq = R$

$P = 1$ $R = 5z + \tan(y-3x)$

$Q = 3$

$$\Rightarrow \frac{dx}{1} = \frac{dy}{3} = \frac{dz}{5z+\tan(y-3x)}$$

I II III

$(I, II) \Rightarrow dx = \frac{dy}{3} \Rightarrow 3dx = dy \Rightarrow 3x = y + C_1 \Rightarrow \boxed{y - 3x = C_1}$

$(I, III) \Rightarrow dx = \frac{dz}{5z + \tan C_1} \Rightarrow x = \frac{1}{5}\log(5z+\tan C_1) + C_2$

$$\boxed{5x - \log[5z + \tan(y-3x)] = C_2}$$

$\int \frac{dz}{5z+\tan C_1}$

$5z + \tan C_1 = t$

$5dz = dt$

$dz = \frac{dt}{5}$

$\int \frac{dt}{5t}$

General sol^n $\phi[y-3x, 5x - \log(5z + \tan(y-3x))] = 0$

Type-II By Using Multiplier

$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$ Aux eq^n $Pp + Qq = R$

let (P_1, Q_1, R_1) (Multipliers)

each fraction $= \frac{P_1dx + Q_1dy + R_1dz}{PP_1 + QQ_1 + RR_1}$ $= 0$

$$\boxed{PP_1 + QQ_1 + RR_1 = 0}$$

each fraction $= \frac{P_1dx + Q_1dy + R_1dz}{0}$

$\Rightarrow \boxed{P_1dx + Q_1dy + R_1dz = 0} \Rightarrow$ sol^n

OR.

(P_1, Q_1, R_1)

each fraction $= \frac{P_1dx + Q_1dy + R_1dz}{PP_1 + QQ_1 + RR_1}$

N^r = exact Diff of D^r

Ex. Solve $(y^2+z^2)p - xyq = -zx$ (AKTU 2016, 2017)

$P\,p + Q\,q = R$

Aux. $\dfrac{dx}{y^2+z^2} = \dfrac{dy}{-xy} = \dfrac{dz}{-zx}$
I II III

(II, III) $\dfrac{dy}{-xy} = \dfrac{dz}{-zx}$

$\Rightarrow \dfrac{dy}{y} = \dfrac{dz}{z}$

$\log y = \log z + \log c_1$

$\Rightarrow \boxed{\dfrac{y}{z} = c_1}$

Multiplier (x, y, z)

each fraction $= \dfrac{xdx + ydy + zdz}{x(y^2+z^2) - xy^2 - z^2x}$

each frac. $= \dfrac{xdx + ydy + zdz}{0}$

$\Rightarrow xdx + ydy + zdz = 0$

$\dfrac{x^2}{2} + \dfrac{y^2}{2} + \dfrac{z^2}{2} = c_2$

$\Rightarrow \boxed{x^2+y^2+z^2 = c_2}$

General sol^n $\phi(u, v) = 0 \Rightarrow \boxed{\phi\left(\dfrac{y}{z}, x^2+y^2+z^2\right) = 0}$

Ex. $\underbrace{(mz-ny)}_{P}p + \underbrace{(nx-lz)}_{Q}q = \underbrace{ly-mx}_{R}$

Aux. eq^n $\dfrac{dx}{mz-ny} = \dfrac{dy}{nx-lz} = \dfrac{dz}{ly-mx}$
I II III

(l, m, n) Each fraction $= \dfrac{ldx + mdy + ndz}{lmz - lny + mnx - lmz + nly - mnx}$

each fraction $= \dfrac{ldx + mdy + ndz}{0}$

$\Rightarrow ldx + mdy + ndz = 0 \Rightarrow \boxed{lx + my + nz = c_1}$

(x, y, z) each fraction $= \dfrac{xdx + ydy + zdz}{mxz - nxy + nxy - lyz + lyz - mxz}$

$= \dfrac{xdx + ydy + zdz}{0}$

$\Rightarrow xdx + ydy + zdz = 0$

$\dfrac{x^2}{2} + \dfrac{y^2}{2} + \dfrac{z^2}{2} = c_2$

$\Rightarrow \boxed{x^2+y^2+z^2 = c_2}$

$\boxed{\phi(lx+my+nz,\ x^2+y^2+z^2) = 0}$

Ex. Solve $\underbrace{y^2(x-y)}_{P}p + \underbrace{x^2(y-x)}_{Q}q = \underbrace{z(x^2+y^2)}_{R}$

$$\text{Aux} = \underset{\text{I}}{\frac{dx}{y^2(x-y)}} = \underset{\text{II}}{\frac{dy}{x^2(y-x)}} = \underset{\text{III}}{\frac{dz}{z(x^2+y^2)}}$$

(I, II) $$\frac{dx}{y^2\cancel{(x-y)}} = \frac{dy}{-x^2\cancel{(x-y)}}$$

$$\Rightarrow y^2dy + x^2dx = 0$$

$$\frac{y^3}{3} + \frac{x^3}{3} = C_1 \Rightarrow \boxed{x^3+y^3 = C_1}$$

(1, -1, 0) each fraction $= \dfrac{dx - dy}{y^2(x-y) - x^2(y-x)}$

$$= \frac{dx-dy}{y^2(x-y)+x^2(x-y)}$$

$$\frac{dx}{y^2(x-y)} = \frac{dy}{x^2(y-x)} = \frac{dz}{z(x^2+y^2)} = \frac{dx-dy}{(x-y)(x^2+y^2)}$$

$$\frac{dz}{z\cancel{(x^2+y^2)}} = \frac{dx-dy}{(x-y)\cancel{(x^2+y^2)}}$$

$$\frac{dz}{z} = \frac{dx-dy}{x-y} = \frac{d(x-y)}{x-y}$$

$$\log z = \log(x-y) + \log C_2$$

$$\log z - \log(x-y) = \log C_2$$

$$\boxed{\frac{z}{x-y} = C_2}$$

$$\phi\left(x^3+y^3, \frac{z}{x-y}\right) = 0$$

L-4 (Unit-1)

Non-Linear PDE of First Order

PDE involves p and q with degree higher than one and product of p & q. → Charpit Method (general Method)

Special Type

Type-I Present only p and q i.e $f(p,q)=0$

$p = \frac{\partial z}{\partial x}$, $q = \frac{\partial z}{\partial y}$.

The complete sol^n is given by $\boxed{z = ax + by + c}$ — (1)

2 Arb. Const = No of indep Var

where a, b are connected by

$$f(a,b) = 0 \quad \text{— (2)}$$

Since $\frac{\partial z}{\partial x} = a = p$, $\frac{\partial z}{\partial y} = b = q$

From eq^n (2), find 'b' in terms of 'a' and Put this Value into eq^n (1)

Ex. Solve $p^2 + q^2 = 2$

$z = ax + by + c$ — (1)

$p \to a$, $q = b$

$a^2 + b^2 = 2 \Rightarrow b^2 = 2 - a^2 \Rightarrow b = \pm\sqrt{2-a^2}$

sol^n $\boxed{z = ax \pm \sqrt{2-a^2}\, y + c}$

Ex. Solve $pq = p + q$ (Type-I) $f(p,q)=0$

$z = ax + by + c$ — (1)

$p \to a$, $q \to b$ $\Rightarrow$ $ab = a + b$

$ab - b = a$

$b(a-1) = a \Rightarrow \boxed{b = \frac{a}{a-1}}$

$$\boxed{z = ax + \frac{a}{a-1} y + c}$$

Type-2 Clairaut Form, $z = px + qy + f(p, q)$

Complete sol^n is given by $z = ax + by + f(a, b)$

by replacing $p \to a$, $q \to b$

Ex. Solve $z = px + qy - 2\sqrt{pq}$ $\quad p \to a,\ q \to b$

$$\boxed{z = ax + by - 2\sqrt{ab}}$$

Ex. Solve $z = px + qy + \sin(p+q)$ $\quad p \to a,\ q \to b$

$$\boxed{z = ax + by + \sin(a+b)}$$

Type-3 Form $f(z, p, q) = 0$ equation not containing x & y

let $z = \phi(x+ay) = \phi(u)$ a trial sol^n

where $u = x + ay$

$$\frac{\partial z}{\partial x} = p = \phi'(x+ay) 1 = \phi'(u) \Rightarrow \boxed{p = \frac{dz}{du}}$$

$$\frac{\partial z}{\partial y} = q = \phi'(x+ay) \cdot a = a\phi'(u) \Rightarrow \boxed{q = a\frac{dz}{du}}$$

⇒ Substitute the value of p, q in given Eq^n.

$\boxed{PDE} \longrightarrow \boxed{\text{Ordinary Diff } Eq^n \text{ in 'z' and 'u'}}$

⇒ Solve ODE

⇒ In last, Replace $u \to x + ay$

Ex. Solve $9(p^2 z + q^2) = 4$. $f(p,q,z)=0$

$z = \phi(x+ay) = \phi(u)$ $u = x+ay$.

$p = \frac{dz}{du}$, $q = a\frac{dz}{du}$

$9\left[\left(\frac{dz}{du}\right)^2 z + a^2\left(\frac{dz}{du}\right)^2\right] = 4$

$\Rightarrow 9\left(\frac{dz}{du}\right)^2 [z+a^2] = 4 \Rightarrow \left(\frac{dz}{du}\right)^2 = \frac{4}{9(z+a^2)}$

$\frac{dz}{du} = \frac{2}{3\sqrt{z+a^2}}$ → O.D.E.

$\Rightarrow du = \frac{3}{2}\sqrt{z+a^2}\,dz$

$\Rightarrow u = \frac{\cancel{3}}{\cancel{2}} \frac{(z+a^2)^{3/2} \cdot \cancel{2}}{\cancel{3}} + b$

$\Rightarrow x+ay = (z+a^2)^{3/2} + b$

$\Rightarrow (x+ay-b) = (z+a^2)^{3/2}$ ←

$\Rightarrow (x+ay-b)^2 = (z+a^2)^3$ ←

Type-4 Equation of form $f_1(x,p) = f_2(y,q)$

As a trial solution

$f_1(x,p) = f_2(y,q) = a$

Solve equations $\boxed{f_1(x,p) = a}$ for p, let $p = F_1(x)$

$\boxed{f_2(y,q) = a}$ for q, let $q = F_2(y)$

we know that $dz = \frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy = p\,dx + q\,dy$

$\boxed{dz = F_1(x)\,dx + F_2(y)\,dy}$

Integrate ↗

L-4-84

Ex. Solve $\underbrace{p-3x^2}_{f_1(p,x)} = \underbrace{q^2-y}_{f_2(q,y)}$

$f_1(p,x) = f_2(q,y)$

(Type-IV)

$$\boxed{p-3x^2 = q^2-y = a}$$

$p-3x^2 = a$

$\Rightarrow \boxed{p = a+3x^2}$

$q^2-y = a$

$\Rightarrow$

$q^2 = a+y$

$q = \sqrt{a+y}$

$$dz = \frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy$$

$$dz = p\,dx + q\,dy$$

$$dz = (a+3x^2)\,dx + \sqrt{a+y}\,dy$$

$$z = \left[ax + \cancel{3}\cdot\frac{x^3}{\cancel{3}}\right] + (a+y)^{3/2}\cdot\frac{2}{3} + b$$

$$z = (ax+x^3) + \frac{2}{3}(a+y)^{3/2} + b$$

L-5 (Unit-1)

Charpit's Method

(General method to solve Non Linear PDE of first order)

$$f(x, y, z, p, q) = 0 \quad \text{———} \; (1)$$

Aux. Eq

$$\frac{dp}{f_x + p f_z} = \frac{dq}{f_y + q f_z} = \frac{dz}{-p f_p - q f_q} = \frac{dx}{-f_p} = \frac{dy}{-f_q}$$

I, II, III, IV, V

$p =$

$q =$

$$\boxed{dz = p\,dx + q\,dy}$$

$f_x = \frac{\partial f}{\partial x}$

$f_z = \frac{\partial f}{\partial z}$

$f_p = \frac{\partial f}{\partial p}$

$f_q = \frac{\partial f}{\partial q}$

Ex solve $px + qy = pq$

$$f \equiv px + qy - pq = 0 \quad \text{———} \; (1)$$

$\frac{\partial f}{\partial x} = f_x = p$ | $f_z = \frac{\partial f}{\partial z} = 0$ | $f_q = \frac{\partial f}{\partial q}$

$\frac{\partial f}{\partial y} = f_y = q$ | $f_p = \frac{\partial f}{\partial p} = x - q$ | $= y - p$

Aux

$$\frac{dp}{f_x + p f_z} = \frac{dq}{f_y + q f_z} = \frac{dz}{-p f_p - q f_q} = \frac{dx}{-f_p} = \frac{dy}{-f_q}$$

$$\frac{dp}{p + p\times 0} = \frac{dq}{q + q\times 0} = \frac{dz}{-p(x-q) - q(y-p)} = \frac{dx}{-(x-q)} = \frac{dy}{-(y-p)}$$

$$\Rightarrow \frac{dp}{p} = \frac{dq}{q} = \frac{dz}{-px + pq - qy + pq} = \frac{dx}{-x+q} = \frac{dy}{-y+p}$$

I, II, III, IV, V

(I^{st}, II)

$$\frac{dp}{p} = \frac{dq}{q} \Rightarrow \log p = \log q + \log a$$

$$\boxed{p = aq} \quad \text{———} \; (2)$$

$Eq^n \quad px + qy - pq = 0$

$$aqx + qy - qa \cdot q = 0 \Rightarrow q(ax+y) = aq^2$$

$$\boxed{q = \frac{ax+y}{a}} \quad \text{--- (3)}$$

$$p = aq = a\left(\frac{ax+y}{a}\right)$$

$$\boxed{p = ax+y} \quad \text{--- (4)}$$

$$\boxed{dz = p\,dx + q\,dy}$$

$$dz = (ax+y)\,dx + \left(\frac{ax+y}{a}\right)dy \leftarrow$$

$$a\,dz = (ax+y)\,a\,dx + (ax+y)\,dy$$

$$a\,dz = (ax+y)\left[a\,dx + dy\right]$$

Let $ax+y = u$

$$a\,dz = u\,du$$

$$\Rightarrow az = \frac{u^2}{2} + b$$

$$\Rightarrow az = \frac{(ax+y)^2}{2} + b$$

$$\Rightarrow 2az = (ax+y)^2 + 2b$$

$$\Rightarrow \boxed{2az = (ax+y)^2 + c}$$

L-6 (Unit-1)

Solution of PDE by using

Cauchy's Method of Characteristics

Consider first order PDE

$$a u_x + b u_y = f(x,y) + ku \quad , \quad \boxed{u(0,y) = g(y)} \quad \text{B.C}$$

$u = u(x,y)$

$a, b,$ depend on x, y and u

$$u_x = \frac{\partial u}{\partial x}, \quad u_y = \frac{\partial u}{\partial y}$$

Aux eqn,

$$\underset{\text{I}}{\frac{dx}{a}} = \underset{\text{II}}{\frac{dy}{b}} = \underset{\text{III}}{\frac{du}{f(x,y)+ku}}$$

Ex. ⇒ Solve by using Cauchy's characteristics Method

$$u_x + u_y = u \quad , \quad \boxed{u(x,0) = 1+e^x} \quad \text{B.C}$$

$$\boxed{y = 0 \ , \ u = 1+e^x}$$

$a u_x + b u_y = f(x,y) + ku$

$a = 1, \ b = 1$

Aux. eqn

$$\frac{dx}{a} = \frac{dy}{b} = \frac{du}{f(x,y)+ku}$$

$$\text{Aux} \Rightarrow \underset{\text{I}}{\frac{dx}{1}} = \underset{\text{II}}{\frac{dy}{1}} = \underset{\text{III}}{\frac{du}{u}} \quad \text{--- (1)}$$

(I, II) $\quad dx = dy \Rightarrow \boxed{x - y = c_1}$ --- (2)

(II, III) $\quad \frac{du}{u} = dy \Rightarrow \log u = y + \log c_2$

$$\Rightarrow \log \frac{u}{c_2} = y$$

$$\frac{u}{c_2} = e^y \Rightarrow u = c_2 e^y \quad \text{--- (3)}$$

c_2

$$\boxed{c_2 = g(c_1)} \Rightarrow c_2 = g(x-y)$$

$$\boxed{u = g(x-y)\,e^{y} \quad (4)} \leftarrow \quad g \to \text{arbi. fn.}$$

$$\boxed{u(x,0) = 1+e^{x}} \longrightarrow y=0,\ u = 1+e^{x}$$

$$1+e^{x} = g(x)\,e^{0} = g(x)$$

$$\boxed{g(x) = 1+e^{x}}$$

$$\boxed{g(x-y) = 1+e^{x-y}}$$

Eq^n (4)

$$u = \left(1+e^{x-y}\right).e^{y}$$

$$u = e^{y} + e^{x}.e^{-y}e^{y}$$

$$\boxed{u = e^{y}+e^{x}} \quad \text{Sol}^n$$

L-7 (Unit-1)

Homogenous Linear PDE with Constant Coefficient

(Complementry Funcion) (C.F.)

Homogenous Linear PDE $\Rightarrow$ All derivaties in PDE are of same order

Ex (1) $\frac{\partial^2 z}{\partial x^2} + 2\frac{\partial^2 z}{\partial y^2} = \sin(xy)$ $\rightarrow$ Homogenous

(2) $\frac{\partial^2 z}{\partial x^2} + 2\frac{\partial^2 z}{\partial y^2} + \frac{\partial z}{\partial x} = \sin(xy)$ $\rightarrow$ Non-homogenous

Let

$$A_0\frac{\partial^n z}{\partial x^n} + A_1\frac{\partial^n z}{\partial x^{n-1}\partial y} + \cdots + A_n\frac{\partial^n z}{\partial y^n} = f(x,y) \quad \text{--- (1)}$$

$A_0, A_1, A_2, \ldots\ldots A_n \Rightarrow$ const

$z \rightarrow$ dep
$x, y \rightarrow$ indep

$\Rightarrow \frac{\partial z}{\partial x} \equiv Dz$ $\quad D \equiv \frac{\partial}{\partial x}$

$\frac{\partial z}{\partial y} = D'z$, $\frac{\partial}{\partial y} \equiv D'$

Convert eq^n into D, D' operators

Eq^n $\boxed{F(D,D')z = f(x,y)}$ --- (2)

Sol^n of $F(D,D') = f(x,y)$ --- (3)

Complete sol^n z = Complementry fn + Particular Integral

$\boxed{z = C.F + P.I.}$

For C.F. General sol^n of $\boxed{F(D,D')z = 0}$

$\Rightarrow$ C.F. must contain arb. fns.

$\boxed{\text{No. of arbitary fns} = \text{Order of P.D.E}}$

Rules of finding C.F.

$$\frac{\partial^2 z}{\partial x^2} + a_1 \frac{\partial^2 z}{\partial x \partial y} + a_2 \frac{\partial^2 z}{\partial y^2} = 0$$

① $F(D, D')z = f(x, y)$

$$D^2 z + a_1 DD'z + a_2 D'^2 z = 0$$

$$(D^2 + a_1 DD' + a_2 D'^2)z = 0 \quad \text{— (A)}$$

for CF.

$$\boxed{F(D, D')z = 0}$$

② Form. Aux. eq^n, $D \to m$, $D' \to 1$, $z \to 1$

Eq^n (A) $\boxed{(m^2 + a_1 m + a_2) = 0} \to$ A.E.

③ Factorize A.E. and find roots of A.E

let roots be m_1, m_2

④ Case I → when roots are distinct

$CF = f_1(y + m_1 x) + f_2(y + m_2 x)$

for ex. $m = 1, 2$

$CF = f_1(y + 1 \cdot x) + f_2(y + 2x)$

Case II when roots are repeated

$m_1 = m_2 = m$

$CF = f_1(y + mx) + x f_2(y + mx)$

---

for $m = 1, 1, 1$

$CF = f_1(y + 1x) + x f_2(y + 1 \cdot x) + x^2 f_3(y + 1 \cdot x)$

Ex. Solve $\frac{\partial^3 z}{\partial x^3} - 4\frac{\partial^3 z}{\partial x^2 \partial y} + 3\frac{\partial^3 z}{\partial x \partial y^2} = 0$

$\frac{\partial}{\partial x} \equiv D$, $\frac{\partial}{\partial y} \equiv D'$

$\Rightarrow (D^3 z - 4D^2D'z + 3D.D'^2 z) = 0$

$(D^3 - 4D^2D' + 3DD'^2)z = 0$ ✓

Complete sol^n $\Rightarrow$ $z = CF + PI$ $\boxed{PI = 0}$

$z = C.F$

<u>For C.F.</u> $D \to m,\ D' \to 1,\ z \to 1$

$m^3 - 4m^2 + 3m = 0$

$m(\underline{m^2 - 4m + 3}) = 0$

$m = 0,\quad m = 3, 1$ (distinct)

$C.F = f_1(y + 0x) + f_2(y + 3x) + f_3(y + 1.x)$

$$\boxed{z = f_1(y) + f_2(y + 3x) + f_3(y + x)}$$

Ex solve $r + 7s + 12t = 0$ ✓

$p = \partial z/\partial x$

$q = \partial z/\partial y$

$r = \partial^2 z/\partial x^2$ ✓

$s = \partial^2 z/\partial x \partial y$ ✓

$t = \partial^2 z/\partial y^2$ ✓

$z = CF + PI$, $PI = 0$

$\boxed{z = CF}$

✓ $(D^2 z + 7DD'z + 12D'^2 z) = 0$

$(D^2 + 7DD' + 12D'^2)z = 0$

A.E $D \to m,\ D' \to 1,\ z \to 1$

$m^2 + 7m + 12 = 0$ $\boxed{m = -4, -3}$

$$\boxed{z = f_1(y - 4x) + f_2(y - 3x)}$$

Ex. $\dfrac{\partial^2 z}{\partial x^2} + a^2 \dfrac{\partial^2 z}{\partial y^2} = 0$ $[PI = 0]$

$\Downarrow$

$(D^2 z + a^2 D'^2 z) = 0$

$(D^2 + a^2 D'^2)z = 0$ $\qquad D \to m,\ D' \to 1,\ z \to 1$

AE $\boxed{m^2 + a^2 = 0}$

$m^2 = -a^2 \Rightarrow \boxed{m = \pm ai}$ (ai, $-ai$)

$$\boxed{z = f_1(y + aix) + f_2(y - aix)}$$

Ex. 1. $\frac{\partial^2 z}{\partial y^2} + 1 \cdot \frac{\partial^2 z}{\partial x \partial y} = 0$ [second order PDE]

$z = CF$ $\qquad \boxed{PI = 0}$

$(D'^2 + DD')z = 0$

$\Rightarrow D'[D' + D]z = 0$

A.E $D \to m,\ D' \to 1,\ z \to 1$

$1 + m = 0$

$m = -1$

Note C.F. corresponding $D' \Rightarrow f_1(x)$

" " $D'^2 \Rightarrow f_1(x) + y f_2(x)$

$D'^3 \Rightarrow f_1(x) + y f_2(x) + y^2 f_3(x)$

CF. corresponding to $D' \Rightarrow f_1(x)$ ✓

$(D + D')z = 0$ $\Rightarrow m + 1 = 0$

$D \to m,\ D' \to 1,\ z \to 1$ $\qquad m = -1$

$C.F \Rightarrow (D + D') \Rightarrow$ $\qquad f_2(y + (-1)x) = f_2(y - x)$

$$\boxed{CF = f_1(x) + f_2(y - x)}$$

Ex. $\frac{\partial^3 z}{\partial x^2 \partial y} - 5 \frac{\partial^3 z}{\partial x \partial y^2} + 6 \frac{\partial^3 z}{\partial y^3} = 0$ (Homogenous)

$D \to m,\ D' \to 1$

$$\left(D^2D' - 5DD'^2 + 6D'^3\right)z = 0$$

$m^2 - 5m + 6 = 0$

② X

$$D'\left[D^2 - 5DD' + 6D'^2\right)z = 0$$

$$D' \Longrightarrow f_1(x)$$

$$\left(D^2 - 5DD' + 6D'^2\right)z = 0$$

$$m^2 - 5m + 6 = 0$$

$$m = 2, 3 \qquad CF \Rightarrow f_2(y+2x) + f_3(y+3x)$$

$$\boxed{z = f_1(x) + f_2(y+2x) + f_3(y+3x)}$$

L-8 (Unit 1)

Homogenous Linear PDE with Constant Coefficient

(Particular Integral)(P.I.)

(CF)

Case-I $f(x,y) = e^{ax+by}$

Case-II $f(x,y) = \sin(ax+by)$ OR $\cos(ax+by)$

$$F(D,D')z = f(x,y) \quad \text{— (1)}$$

$$\boxed{z = C.F + P.I.}$$

Rules of P.I.

$$\boxed{P.I = \frac{1}{F(D,D')} f(x,y)}$$

Case I $\quad f(x,y) = e^{ax+by}$ $\qquad D \equiv \frac{\partial}{\partial x}$, $D' \equiv \partial/\partial y$

$$PI = \frac{1}{F(D,D')} e^{ax+by}$$

$(D \to a, D' \to b)$

$$= \frac{1}{F(a,b)} e^{ax+by}, \quad F(a,b) \neq 0$$

If $F(a,b) = 0$ $\qquad$ Case failure

$$P.I = x \frac{1}{\frac{\partial}{\partial D} F(D,D')} e^{ax+by}, \quad D \to a, \; D' \to b$$

OR

$$= y \frac{1}{\frac{\partial}{\partial D'} F(D,D')} e^{ax+by}, \quad D \to a, \; D' \to b$$

Ex. Solve $(D^2 + 2DD' + D'^2)z = e^{2x+3y}$ (2015) (Second PDE)

$\boxed{z = CF + PI}$

Sol Complete sol $z = CF + PI$

For CF $\quad (D^2+2DD'+D'^2)z=0 \qquad D\to m,\ D'\to 1,\ z\to 1$

A.E $\quad m^2+2m+1=0$

$(m+1)^2=0$

$m=-1,-1$ (Repeated Roots)

$CF = f_1(y+(-1)x) + x f_2(y+(-1)x)$

$$\boxed{CF \quad f_1(y-x)+x f_2(y-x)}$$

$PI = \frac{1}{F(D,D')} e^{ax+by}$

$= \frac{1}{D^2+2DD'+D'^2} e^{2x+3y} \qquad D\to 2,\ D'\to 3$

$= \frac{1}{4+2\times2\times3+9} e^{2x+3y}$

$= \frac{1}{25} e^{2x+3y}$

$z = f_1(y-x)+x f_2(y-x)+\frac{1}{25}e^{2x+3y}$ //

Ex Solve $\quad (D^2-4DD'+4D'^2)z = e^{2x+y}$

$$\boxed{z = CF+PI}$$

A.E. $\quad m^2-4m+4=0$

$(m-2)^2=0 \Rightarrow m=2,2$

$$\boxed{CF = f_1(y+2x)+x f_2(y+2x)}$$

$PI = \frac{1}{D^2-4DD'+4D'^2} e^{2x+y} \qquad D\to 2,\ D'\to 1$

$= \frac{1}{4-4\times2\times1+4} e^{2x+y}$

$= \frac{1}{0}$ (case failure)

$$PI = x \frac{1}{\frac{\partial}{\partial D}[D^2 - 4DD' + 4D'^2]} e^{2x+y}$$

$$= x \frac{1}{2D - 4D'} e^{2x+y}$$

$D \to 2$
$D' \to 1$

$$= x \frac{1}{2\times 2 - 4\times 1} e^{2x+y}$$ (case failure)

$$= x^2 \frac{1}{\frac{\partial}{\partial D}(2D - 4D')} e^{2x+y}$$

$$= x^2 \frac{1}{2} e^{2x+y} = \frac{x^2}{2} e^{2x+y}$$

$$\boxed{Z = f_1(y+2x) + x f_2(y+2x) + \frac{x^2}{2} e^{2x+y}}$$

Ex Solve $(D^2 + 7DD' + 12D'^2)Z = \sinh x$ [2017]

$\sinh x = \frac{e^x - e^{-x}}{2}$

$\cosh x = \frac{e^x + e^{-x}}{2}$

CF $\Rightarrow$

A.E $\Rightarrow m^2 + 7m + 12 = 0$

$m = 3, 4$

$$\boxed{CF = f_1(y+3x) + f_2(y+4x)}$$

$$PI = \frac{1}{D^2 + 7DD' + 12D'^2} \sinh x$$

$$= \frac{1}{\ldots} \left[\frac{e^x - e^{-x}}{2}\right]$$

$$D^2+7DD'+12D'^2$$

$$= \frac{1}{2}\left[\frac{1}{D^2+7DD'+12D'^2}e^{x} - \frac{1}{D^2+7DD'+12D'^2}e^{-x}\right]$$

$D \to 1$, $D' \to 0$; $D \to -1$, $D' \to 0$

$$\frac{1}{2}\left[\frac{1}{1}e^{x} - \frac{1}{1}e^{-x}\right] = \frac{e^{x}-e^{-x}}{2}.$$

$$= \sinh x$$

$$Z = CF + PI$$

$$Z = f_1(y-3x) + f_2(y-4x) + \sinh x$$

Case $f(x,y) = \sin(ax+by)$ / $\cos(ax+by)$

$$PI = \frac{1}{F(D^2, DD', D'^2)}\sin(ax+by)$$

$$= \frac{1}{F(-a^2,-ab,-b^2)}\sin(ax+by);$$

$D^2 \to -a^2$

$D'^2 \to -b^2$

$DD' \to -(a.b)$

$$F(-a^2,-ab,-b^2) \neq 0$$

Solve $r+s-6t = \cos(2x+y)$

$r = \frac{\partial^2 z}{\partial x^2} = D^2 z$

$s = \frac{\partial^2 z}{\partial x \partial y} = DD'z$

$t = \frac{\partial^2 z}{\partial y^2} = D'^2 z$

$$(D^2 + DD' - 6D'^2)z = \cos(2x+y)$$

$D \to m$, $D' \to 1$, $z \to 1$

CF

A.E.

$$m^2 + m - 6 = 0$$

$m = 2, -3$

$m = 2, -3$

$$\boxed{CF = f_1(y+2x) + f_2(y-3x)}$$

$$PI = \frac{1}{D^2 + DD' - 6D'^2} \cos(2x+y)$$

$D^2 \to -2^2$

$D^2 \to -4$

$DD' \to -(2\times 1)$

$D'^2 \to -(1^2)$

$D'^2 \to -1$

$$= \frac{1}{-4-2-6(-1)} \cos(2x+y)$$

$\frac{1}{0}$ (case failure)

$$PI = x\left[\frac{1}{2D+D'} \cos(2x+y)\right]$$

$$= x\left[\frac{(2D-D')}{(2D+D')(2D-D')} \cos(2x+y)\right]$$

$$= x\left[\frac{2D-D'}{4D^2-D'^2} \cos(2x+y)\right]$$

$D^2 \to -4$

$D'^2 \to -1$

$$= x\left[\frac{2D-D'}{4(-4)-(-1)} \cos(2x+y)\right]$$

$D \equiv \frac{\partial}{\partial x}$

$D' \equiv \partial/\partial y$

$$= x\left[\frac{(2D-D')}{-15} \cos(2x+y)\right]$$

$$= -\frac{x}{15}\left[2D\cos(2x+y) - D'\cos(2x+y)\right]$$

$$= -\frac{x}{15}\left[-2.\sin(2x+y).2 + \sin(2x+y)1\right]$$

$$= -\frac{x}{15}\left[-4+1\right]\sin(2x+y)$$

$$= \frac{x}{\cancel{15}_{5}} . \cancel{3} \sin(2x+y)$$

$$= \frac{x}{5}\sin(2x+y)$$

$$= \frac{x}{5}\sin(2x+y)$$

$$Z = CF + PI$$
$$= f_1(y+2x) + f_2(y-3x) + \frac{x}{5}\sin(2x+y)$$

Ex. find Particular Integral of

$$\frac{\partial^2 z}{\partial x^2} - \frac{\partial^2 z}{\partial x \partial y} = \sin x \cos 2y \quad [2018]$$

$$(D^2 - DD')z = \sin x \cos 2y$$

$$PI = \frac{1}{2(D^2 - DD')} 2\sin x \cos 2y$$

$$2\sin A \cos B = \sin(A+B) + \sin(A-B)$$

$$= \frac{1}{2}\left[\frac{1}{D^2 - DD'}\left\{\sin(x+2y) + \sin(x-2y)\right\}\right]$$

$$= \frac{1}{2}\left[\frac{1}{D^2-DD'}\sin(x+2y) + \frac{1}{D^2-DD'}\sin(x-2y)\right]$$

$D^2 \rightarrow -1$
$DD' \rightarrow -(1.2)$

$D^2 \rightarrow -1$
$DD' \rightarrow -(1.(-2))$
$DD' \rightarrow 2$

$$\frac{1}{2}\left[\frac{1}{-1+2}\sin(x+2y) + \frac{1}{-1-2}\sin(x-2y)\right]$$

$$= \frac{1}{2}\left[\sin(x+2y) + \frac{1}{(-3)}\sin(x-2y)\right]$$

$$\boxed{P.I. = \frac{1}{2}\sin(x+2y) - \frac{1}{6}\sin(x-2y)}$$

Long → 10

L-9 (Unit -1)

Homogenous Linear PDE with Const. Coeff.

(Particular Integral) Part-2

Case III ⇒ $f(x,y) = x^m y^n$

Case IV ⇒ $f(x,y) = \phi(ax+by)$

$$\boxed{F(D,D')z = f(x,y)}$$

$z = CF + P.I.$

L-7 → C.F.

L-8, 9, 10] → P.I.

Case III $\quad f(x,y) = x^m y^n$

$$PI = \frac{1}{F(D,D')} x^m y^n$$

$$= \left[1 \pm \phi(D,D')\right]^{-1} x^m y^n$$

$(1+x)^{-1} = 1 - x + x^2 - x^3 \ldots$

$(1-x)^{-1} = 1 + x + x^2 \ldots$

$(1+x)^{-2} = 1 - 2x + 3x^2 - 4x^3 \ldots$

Note :- (i) $m > n$, exp. in ascending power $\frac{D'}{D}$

(ii) $n > m$, exp in ascending power $\frac{D}{D'}$

Ex. Solve $\frac{\partial^3 z}{\partial x^3} - 2\frac{\partial^3 z}{\partial x^2 \partial y} = 3x^2 y$

$(D^3 - 2D^2D')z = 3x^2 y$

$z = CF + P.I.$

CF ⇒ $D \to m,\ D' \to 1,\ z \to 1$

$m^3 - 2m^2 = 0$

$m^2(m-2) = 0 \quad \boxed{m = 0, 0, 2}$

$CF = f_1(y+0x) + x f_2(y+0x) + f_3(y+2x)$

$$PI = \frac{1}{D^3 - 2D^2D'} 3x^2 y$$

$$= \frac{3}{D^3}\left[\frac{1}{1-2\frac{D'}{D}}\right] x^2 y$$

$$= \frac{3}{D^3}\left[1-2\underbrace{\frac{D'}{D}}\right]^{-1} x^2 y$$

$$= \frac{3}{D^3}\left[1+\left(\frac{2D'}{D}\right)+\left(\frac{2D'}{D}\right)^2 \cdots\right] x^2 y'$$

$$= \frac{3}{D^3}\left[1+\frac{2D'}{D}\right] x^2 y \Rightarrow \frac{3}{D^3}\left[x^2 y + \frac{2}{D} D'(x^2 y)\right]$$

$$= \frac{3}{D^3}\left[x^2 y + \frac{2}{D}(x^2)\right]$$

$$= \frac{3}{D^3}\left[x^2 y + 2 \cdot \frac{x^3}{3}\right]$$

$$= \frac{3}{D^2}\left[\frac{x^3}{3} \cdot y + \frac{2}{3} \cdot \frac{x^4}{4}\right]$$

$$= \frac{3}{D}\left[\frac{y}{3} \cdot \frac{x^4}{4} + \frac{2}{12} \cdot \frac{x^5}{5}\right]$$

$$= 3\left[\frac{y}{12} \cdot \frac{x^5}{5} + \frac{\cancel{2}}{12 \times 5} \cdot \frac{x^6}{\cancel{6}\,3}\right]$$

$$= 3\left[\frac{y x^5}{60} + \frac{x^6}{180}\right]$$

$\frac{D'}{D}$

$\frac{\partial}{\partial y} \equiv D'$, $D = \frac{\partial}{\partial x}$

D'

$D = \partial/\partial x$

$\frac{1}{D} \cdots = \int \cdots dx$

$\frac{1}{D'} = \int \cdots dy$

Gar 3

$$\boxed{Z = CF + PI}$$

Case IV $\quad f(x,y) = \phi(ax+by)$

$$PI = \frac{1}{F(D, D')} \phi(ax+by)$$

$D \to a$

$\tan(x+y)$

$\log(x+2y)$

$\sqrt{x+y}$

$$D' \to b$$

$$= \frac{1}{F(a,b)} \iint\cdots\int \phi(u)\,du\,du\cdots du, \qquad ax+by \to u$$

$$F(a,b) \neq 0$$

Ex. solve $(D^2 + DD' - 2D'^2)Z = \sqrt{2x+y}$ gmp

OR $(r + s - 2t) = (2x+y)^{1/2}$

$\phi(ax+by) = \sqrt{2x+y}$

$a=2 \quad b=1$

$Z = CF + PI$

CF $\Rightarrow$ AE $m^2 + m - 2 = 0$

$m = 1, -2$ (distinct)

$CF = f_1(y + 1x) + f_2(y + (-2x))$

$$\boxed{CF = f_1(y+x) + f_2(y-2x)}$$

$\phi(ax+by) = \sqrt{2x+y}$

$a=2,\ b=1$

$D \to a \to 2$

$D' \to b \to 1$

$2x+y \to u$

$$PI = \frac{1}{D^2 + DD' - 2D'^2}\sqrt{2x+y}$$

$$= \frac{1}{4 + \cancel{2} - \cancel{2 \times 1}} \iint \sqrt{u}\,du\,du$$

$$= \frac{1}{4}\int u^{3/2} \cdot \frac{2}{3}\,du$$

$$= \frac{1}{\cancel{4}}\,\frac{\cancel{2}}{3} \cdot u^{5/2} \cdot \frac{\cancel{2}}{5} = \frac{u^{5/2}}{15} = \frac{(2x+y)^{5/2}}{15}$$

$Z = CF + PI$

Ex. Solve $(D-D')^2 z = \tan(x+y)$

OR

find P.I. of $(D-D')^2 z = \tan(x+y)$

$CF \Rightarrow$ A.E $\Rightarrow (m-1)^2 = 0$

$m = 1, 1$

$$\boxed{CF = f_1(y+x) + x f_2(y+x)}$$

$$PI = \frac{1}{(D-D')^2} \tan(x+y)$$

$\phi(ax+by)$

$= \tan(x+y)$

$a=1, \ b=1$

$\frac{1}{0}$ (Case failure)

$D \to a \to 1$

$D' \to b \to 1$

$$= x\left[\frac{1}{\frac{\partial}{\partial D}(D-D')^2} \tan(x+y)\right]$$

$$= x\left[\frac{1}{2(D-D')} \tan(x+y)\right]$$

$D \to 1$

$D' \to 1$

$$= x^2\left[\frac{1}{2} \tan(x+y)\right]$$

$$PI = \frac{x^2}{2} \tan(x+y)$$

$$\boxed{z = CF + P.I.}$$

L-10 (Unit-1)

Homogenous Linear PDE with Const. Coeff.

Particular Integral(Part-3)

General Method

$$\frac{1}{D-mD'}\phi(x,y) = \int \phi(x, c-mx)\,dx$$

$\boxed{y=c-mx}$

L-8 PI
e^{ax+by},
$\sin(ax+by)$
L-9
$x^m y^n$,
$\phi(ax+by)$

$$PI = \frac{1}{F(D,D')}\phi(x,y)$$

Ex. $(D^2+DD'-6D'^2)z = y\sin x$ (2011, 2012.)..

$z = CF + PI$

$CF \Rightarrow$ A.E $\Rightarrow m^2+m-6=0$

$(m+3)(m-2)=0 \Rightarrow \boxed{m=-3,2}$

$$\boxed{CF \Rightarrow f_1(y-3x) + f_2(y+2x)}$$

(General Method)

$D^2+DD'-6D'^2$
$(D+3D')(D-2D')$ | $D\to m$, $D'\to 1$

$$PI = \frac{1}{D^2+DD'-6D'^2}\, y\sin x$$

$$= \frac{1}{(D+3D')(D-2D')}\, y\sin x$$

$$= \frac{1}{(D+3D')}\left[\frac{1}{D-2D'}\, y\sin x\right]$$

$D-2D'$
$y = c-mx$
$\boxed{y = c-2x}$

$$= \frac{1}{D+3D'}\left[\int \underset{I}{(c-2x)}\,\underset{II}{\sin x}\,dx\right]$$

ILATE

$\int \underset{I}{u}\cdot\underset{II}{v}\,dx$

$= \frac{1}{...}\left[(c-2x)(-\cos x) - \int (+2)(+\cos x)\,dx\right]$

$u\int v\,dx - \int\{d(u)\int v\}$

$$= \frac{1}{D+3D'}\left[(c-3x)(\ldots)\right]$$

$$= \frac{1}{D+3D'}\left[-y\cos x - 2\sin x\right]$$

$$= -\int \underset{\mathrm{I}}{(c+3x)}\,\underset{\mathrm{II}}{\cos x}\,dx - 2\int \sin x\,dx$$

$$= -\left[(c+3x)(\sin x) - \int 3\sin x\,dx\right] + 2\cos x$$

$$= -\left[y\sin x + 3\cos x\right] + 2\cos x$$

$$= -y\sin x - 3\cos x + 2\cos x$$

$$PI = -y\sin x - \cos x$$

$D+3D'$

$y = c - mx$

$c-(-3)x$

$\boxed{y = c+3x}$

$$Z = CF + PI$$

$$\boxed{Z = f_1(y-3x) + f_2(y+2x) + (-y\sin x) - \cos x}$$

Ex $\rightarrow$ $(D^2 + DD' - 6D'^2)Z = y\cos x$ Imp

Ex. $\Rightarrow$ $(D^2 - DD' - 2D'^2)Z = (y-1)e^x$

Topic:→ Non-Homogenous Linear PDE With Constant coefficient (10)

When All partial derivatives in PDE are not of same order.

Ex. (1) $\frac{\partial^2 z}{\partial x^2} - \frac{\partial^2 z}{\partial y^2} + \frac{\partial z}{\partial x} - \frac{\partial z}{\partial y} = 0$ (Non-homog)

(2) $\frac{\partial^3 z}{\partial x^3} - 3\frac{\partial^2 z}{\partial x \partial y} + \frac{\partial z}{\partial y} + 4z = e^{2x+y}$ (Non-homog)

Rules for finding Solution:

Step-1 $\frac{\partial}{\partial x} \equiv D$, $\frac{\partial}{\partial y} \equiv D'$

Ex→1 $(D^2 - D'^2 + D - D')z = 0$

In general. $\boxed{\phi(D, D')z = f(x,y)}$ → (1)

Complete soln $\boxed{z = C.F. + P.I}$

Methods for finding C.F.

Step-1 Resolve $\phi(D, D')z = 0$ into linear factors like $(D - mD' - a)$

Then C.F. corresponding to $(D - mD' - a)$ is
$$e^{-\frac{(-a)}{1}x} f_1(y + mx) = e^{ax} f_1(y+mx)$$

C.F. corresponding to $(D - mD' - a)^2$ is.
$$= e^{-\frac{(-a)}{1}x} f_1(y+mx) + x e^{ax} f_2(y+mx)$$

C.F corresponding to $(D + mD' + a) \Rightarrow e^{-ax} f_1(y - mx)$

C.F. corresponding to $(\alpha D + \beta D' + \gamma) = e^{-\left(\frac{\gamma}{\alpha}\right)x} f_1(\alpha y - \beta x)$

C.f " " $(\alpha D + \gamma) = e^{-\left(\frac{\gamma}{\alpha}\right)x} f_1(\alpha y)$

C.f " " $(\beta D' + \gamma) = e^{-\left(\frac{\gamma}{\beta}\right)y} f_1(\beta x)$

Ex. Solve $r - t + p - q = 0$

$$\frac{\partial^2 z}{\partial x^2} - \frac{\partial^2 z}{\partial y^2} + \frac{\partial z}{\partial x} - \frac{\partial z}{\partial y} = 0$$

$$(D^2 - D'^2 + D - D')z = 0$$

$$\Rightarrow [(D-D')(D+D') + (D-D')]z = 0$$

$$\boxed{(D-D')(D+D'+1)z = 0}$$

$z = CF + PI$

$PI = 0$

$\boxed{z = C.F}$

$D - D' \Rightarrow e^{0x} f_1(y+x)$

$(D+D'+1) = e^{-1x} f_2(y-x)$

$$\boxed{z = e^{0x} f_1(y+x) + e^{-x} f_2(y-x)}$$

$p = \frac{\partial z}{\partial x}$, $q = \frac{\partial z}{\partial y}$, $r = \frac{\partial^2 z}{\partial x^2}$, $s = \frac{\partial^2 z}{\partial x \partial y}$, $t = \frac{\partial^2 z}{\partial y^2}$

Ex. Solve $(D + 4D' + 5)^2 z = 0$

$\boxed{z = CF}$

$(D + 4D' + 5) \Rightarrow e^{-5x} f_1(y - 4x)$

$$z = e^{-5x} f_1(y-4x) + x e^{-5x} f_2(y-4x)$$

Ex. Solve $DD'(2D + D' + 1)z = 0$

C.F. corresponding to $D \Rightarrow e^{0x} f_1(y)$

C.F " $D' \Rightarrow e^{0 \cdot y} f_2(x)$

C.F " " $(2D + D' + 1) \Rightarrow e^{-\left(\frac{1}{2}\right)x} f_3(2y - x)$

$$z = f_1(y) + f_2(x) + e^{-\frac{x}{2}} f_3(2y - x)$$

Ex. Solve $(D^2 + D'^2 - p^2)z = 0$

$D^2 + D'^2 - p^2$ cannot be resolved into linear factors.

let trial soln be $z = Ae^{hx+ky}$, $A, h, k \rightarrow$ const

$$(D^2 + D'^2 - p^2) Ae^{hx+ky} = 0$$

$$D^2(Ae^{hx+ky}) + D'^2(Ae^{hx+ky}) - p^2 Ae^{hx+ky} = 0$$

$$\Rightarrow Ah^2 e^{hx+ky} + Ak^2 e^{hx+ky} - p^2 A e^{hx+ky} = 0$$

$$\Rightarrow Ae^{hx+ky}[h^2 + k^2 - p^2] = 0$$

$Ae^{hx+ky} \neq 0$, $\boxed{h^2 + k^2 - p^2 = 0}$

$D \equiv \frac{\partial}{\partial x}$, $D' \equiv \frac{\partial}{\partial y}$

$D(Ae^{hx+ky})$

$Ae^{hx+ky} h$

$D^2(Ae^{hx+ky})$

$= Ae^{hx+ky} h^2$

$$z = CF = \sum Ae^{hx+ky}, \text{ when } h^2 + k^2 - p^2 = 0$$

Topic :→ Particular Integral of Non-Homogenous Linear PDE with const. coeff.

$\phi(D, D')z = F(x,y)$

Case I $F(x,y) = e^{ax+by}$

Case II ⇒ $F(x,y) = \sin(ax+by)$ OR $\cos(ax+by)$

Case I $F(x,y) = e^{ax+by}$

$PI = \frac{1}{\phi(D,D')} e^{ax+by}$, $[D \to a,\ D' \to b]$

$= \frac{1}{\phi(a,b)} e^{ax+by}$, $\phi(a,b) \neq 0$

Ex. Solve $(D^2 - 4DD' + 4D'^2 - D + 2D')Z = e^{3x+4y}$ (Non homog) (2013)

$Z = CF + PI$

For C.F., $(D^2 - 4DD' + 4D'^2 - D + 2D')Z = 0$

$[(D-2D')^2 - 1(D-2D')]Z = 0$

$(D-2D')[D-2D'-1]Z = 0$

C.F. corresponding to $D - 2D' \Rightarrow e^{0x} f_1(y+2x)$

C.F. " " $(D-2D'-1) \Rightarrow e^{-(-1)x} f_2(y+2x)$

$CF = f_1(y+2x) + e^{x} f_2(y+2x)$

For P.I. ⇒ $PI = \frac{1}{D^2 - 4DD' + 4D'^2 - D + 2D'} e^{3x+4y}$ $\Big|\ D \to 3,\ D' \to 4$

$= \frac{1}{9 - 4(3)(4) + 4 \times 16 - 3 + 2 \times 4} e^{3x+4y}$

$= \frac{1}{9 - 48 + 64 - 3 + 8} e^{3x+4y}$

$= \frac{1}{81-51} e^{3x+4y} = \frac{1}{30} e^{3x+4y}$

$\boxed{Z = CF + PI}$

Case-II $F(x,y) = \sin(ax+by)$ OR $\cos(ax+by)$

$$PI = \frac{1}{\phi(D^2, DD', D'^2)} \sin(ax+by)$$

$$= \frac{1}{\phi(-a^2, -ab, -b^2)} \sin(ax+by)$$

$D^2 \to -a^2$
$D'^2 \to -b^2$
$DD' \to -(a)(b)$

Ex. Solve $(D+1)(D+D'-1)Z = \sin(2x+3y)$ (2019) (10)

C.F. Corresponding $(D+1) \Rightarrow e^{-1x} f_1(y)$

C.F " $(D+D'-1) \Rightarrow e^{-(-1)x} f_2(y-x)$

$$CF = e^{-x} f_1(y) + e^{x} f_2(y-x)$$

$$PI = \frac{1}{(D+1)(D+D'-1)} \sin(2x+3y) = \frac{1}{D^2+DD'-\cancel{D}+\cancel{D}+D'-1} \sin(2x+3y)$$

$$= \frac{1}{D^2+DD'+D'-1} \sin(2x+3y)$$

$D^2 \to -2^2 = -4$
$DD' \to -(2)(3) = -6$
$D'^2 \to -9$

$$= \frac{1}{-4-6+D'-1} \sin(2x+3y)$$

$$= \frac{1}{D'-11} \sin(2x+3y)$$

$$= \frac{D'+11}{(D'-11)(D'+11)} \sin(2x+3y) = \frac{D'+11}{D'^2-121} \sin(2x+3y)$$

$$= \frac{D'+11}{-9-121} \sin(2x+3y) = \frac{1}{-130} (D'+11) \sin(2x+3y)$$

$$= -\frac{1}{130} \left[D' \sin(2x+3y) + 11 \sin(2x+3y) \right]$$

$$= -\frac{1}{130} \left[\frac{\partial}{\partial y} \sin(2x+3y) + 11 \sin(2x+3y) \right]$$

$$= -\frac{1}{130} \left[3\cos(2x+3y) + 11 \sin(2x+3y) \right]$$

$$\boxed{Z = CF + PI}$$

Ex $(D-D'-1)(D-D'-2)Z = \sin(2x+3y)$ ✓

Topic: Particular Integral of Non-Homogenous Linear PDE with Const. Coeff.

$\begin{bmatrix} I & e^{ax+by} \\ II & \sin(ax+by) \end{bmatrix}$

Case III $F(x,y) = x^m y^n$.

Case IV $F(x,y) = e^{ax+by} V(x,y)$

Case III $PI = \frac{1}{\phi(D,D')} x^m y^n = [1 \pm f(D,D')]^{-1} x^m y^n$

Can be expanded in ascending power of $\frac{D'}{D}$ $(m>n)$ or $\frac{D}{D'}$ $(n>m)$ or D or D'

(10)

Ex. solve $D(D+D'-1)(D+3D'-2)z = x^2 - 4xy + 2y^2$. (2020)

C.F. corresponding to $D \Rightarrow e^{0x} f_1(y)$

" " " $(D+D'-1) \Rightarrow e^{-(-1)x} f_2(y-x)$

" " " $(D+3D'-2) \Rightarrow e^{-(-2)x} f_3(y-3x)$

$$CF = f_1(y) + e^{x} f_2(y-x) + e^{2x} f_3(y-3x)$$

$$PI = \frac{1}{D(D+D'-1)(D+3D'-2)} (x^2 - 4xy + 2y^2)$$

$$= \frac{1}{2D[1-(D+D')][1 - \frac{D+3D'}{2}]} (x^2-4xy+2y^2)$$

$(1-x)^{-1} = 1+x+x^2 - - -$

$$= \frac{1}{2D} [1-(D+D')]^{-1} \left[1 - \left(\frac{D+3D'}{2}\right)\right]^{-1} (x^2-4xy+2y^2)$$

$$= \frac{1}{2D} \left[1 + D + D' + D^2 + D'^2 + 2DD' - - -\right]\left[1 + \frac{D}{2} + \frac{3D'}{2} + \frac{D^2}{4} + \frac{9D'^2}{4} + \frac{6DD'}{4} ...\right] (x^2-4xy+2y^2)$$

$$= \frac{1}{2D}\Big[1 + \left(\frac{D}{2} + D\right) + \left(\frac{3D'}{2} + D'\right) + \left(\frac{D^2}{4} + \frac{D^2}{2} + D^2\right) + \left(\frac{9D'^2}{4} + \frac{3}{2}D'^2 + D'^2\right) + \left(\frac{3}{2}DD' + \frac{3}{2}DD' + \frac{1}{2}DD' + 2DD'\right)\Big] (x^2-4xy+2y^2)$$

$$= \frac{1}{2D}\left[1 + \frac{3D}{2} + \frac{5D'}{2} + \frac{7D^2}{4} + \frac{19}{4}D'^2 + \frac{11}{2}DD'\right](x^2-4xy+2y^2)$$

$$\frac{1}{2D}\left[x^2 - 4xy + 2y^2 + \frac{3}{2}(2x-4y) + \frac{5}{2}(-4x+4y) + \frac{7}{4}(2) + \frac{19}{4}(4) + \frac{11}{2}(-4)\right]$$

$$\frac{1}{2D}\left[x^2-4xy+2y^2+3(x-2y)+5(-2x+2y)+\frac{7}{2}+19-22\right]$$

$$\frac{1}{2D}\left[x^2-4xy+2y^2-7x+4y+\frac{1}{2}\right] \quad D=\frac{\partial}{\partial x}$$

$$=\frac{1}{2}\left[\frac{x^3}{3}-4\frac{x^2}{2}y+2y^2x-7\frac{x^2}{2}+4yx+\frac{x}{2}\right]$$

$$\boxed{Z = CF + PI.}$$

Case IV

$$F(x,y) = e^{ax+by}\,V(x,y)$$

$$PI = \frac{1}{\phi(D,D')}\,e^{ax+by}\,V(x,y)$$

$$PI = e^{ax+by}\left[\frac{1}{\phi(D+a,\,D'+b)}\,V(x,y)\right]$$

$D \to D+a,$
$D' \to D'+b$

Ex. solve. $(D-3D'-2)^2 z = 2e^{2x}\tan(y+3x)$

$$CF = e^{-(-2)x}f_1(y+3x) + x\,e^{2x}f_2(y+3x)$$

$$PI = \frac{1}{(D-3D'-2)^2}\,2e^{2x}\tan(y+3x)$$

$$= 2e^{2x}\left[\frac{1}{(D+\cancel{2}-3D'-\cancel{2})^2}\tan(y+3x)\right]$$

$e^{ax+by}V$
$e^{2x}\tan(y+3x)$
$a=2 \quad b=0$
$D \to D+2$
$D' \to D'+0$

$$= 2e^{2x}\left[\frac{1}{(D-3D')^2}\tan(y+3x)\right]$$

$\phi(ax+by) = \tan(y+3x)$

$D \to a \quad a=3$

$D' \to b \quad b=1$

$$= 2e^{2x}\left[\frac{1}{(3-3.1)^2}\tan(y+3x)\right]$$

Case failure

$$= 2e^{2x}\,x\left[\frac{1}{2(D-3D')}\tan(y+3x)\right]$$ Case failure

$$= \cancel{2}e^{2x}\,x^2\left[\frac{1}{\cancel{2}.1}\tan(y+3x)\right]$$

$$PI = x^2 e^{2x}\tan(y+3x)$$

$Z = CF + PI.$

Case I $= e^{ax+by}$

Case II $= \sin(ax+by)$ or $\cos(ax+by)$

III $\Rightarrow x^m y^n.$

IV $= e^{ax+by}\,V(x,y)$

L-14 (Unit-1)

Topic:- Eqⁿ Reducible to PDE with Const. Coefficient

Type:→ Ex.

$$x^3\frac{\partial^3 z}{\partial x^3} - 4xy\frac{\partial^2 z}{\partial x\partial y} + 4y^2\frac{\partial^2 z}{\partial y^2} + x\frac{\partial z}{\partial x} + 6y\frac{\partial z}{\partial y} = x^3y^4$$

An eq^n in which coeff of derivative of order 'k' is multiple of the Variable of degree 'k'

Step-1 Put $x = e^u \Rightarrow u = \log x \Rightarrow \frac{\partial u}{\partial x} = \frac{1}{x}$

$y = e^v \Rightarrow v = \log y$

Step-2 $\frac{\partial z}{\partial x} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial x} = \frac{\partial z}{\partial u}\cdot\frac{1}{x}$

$\Rightarrow x\frac{\partial z}{\partial x} = \frac{\partial z}{\partial u}$, Now $x\frac{\partial z}{\partial x} = Dz$, $D \equiv \frac{\partial}{\partial u}$

Similarly $x^2\frac{\partial^2 z}{\partial x^2} = D(D-1)z$

$x^3\frac{\partial^3 z}{\partial x^3} = D(D-1)(D-2)z$

$y\frac{\partial z}{\partial y} = D'z$, where $D' \equiv \frac{\partial}{\partial v}$

$y^2\frac{\partial^2 z}{\partial y^2} = D'(D'-1)z$

$xy\frac{\partial^2 z}{\partial x\partial y} = DD'z$

Ques $\Longrightarrow$ PDE with const coeff

$z = CF + PI$

$z \Rightarrow$ in terms ~~of~~ u & v.

Ex. Solve $(x^2D^2 + 2xy\,DD' + y^2D'^2)z = x^m y^n$ (2020)

$$x^2\frac{\partial^2 z}{\partial x^2} + 2xy\frac{\partial^2 z}{\partial x\partial y} + y^2\frac{\partial^2 z}{\partial y^2} = x^m y^n.$$

(Reducible)

Put $x = e^u \Rightarrow u = \log x$
$y = e^v \Rightarrow v = \log y$

$x\frac{\partial z}{\partial x} = Dz \Rightarrow D \equiv \frac{\partial}{\partial u}$, $y\frac{\partial z}{\partial y} = D'z$, $D' \equiv \frac{\partial}{\partial v}$

$x^2\frac{\partial^2 z}{\partial x^2} = D(D-1)z$ ✓, $y^2\frac{\partial^2 z}{\partial y^2} = D'(D'-1)z$ ✓

$xy\frac{\partial^2 z}{\partial x\partial y} = DD'z$ ✓

$$D(D-1)z + 2DD'z + D'(D'-1)z = (e^u)^m (e^v)^n.$$

$$\Rightarrow [D^2 - D + 2DD' + D'^2 - D']z = e^{mu+nv}$$

$$\Rightarrow [(D+D')^2 - (D+D')]z = e^{mu+nv}$$

$$\Rightarrow (D+D')(D+D'-1)z = e^{mu+nv} \quad [\text{Non-Homog}]$$

$z = CF + PI$

CF. C.F. corresponding to $D+D' = e^{0u} f_1(v-u)$
C.F. " " $(D+D'-1) = e^{-(-1)u} f_2(v-u)$

$$CF = f_1(v-u) + e^u f_2(v-u)$$
$$= f_1(\log y - \log x) + x f_2(\log y - \log x)$$
$$\boxed{= f_1\left(\log\frac{y}{x}\right) + x f_2\left(\log\frac{y}{x}\right)}$$

$$PI = \frac{1}{(D+D')(D+D'-1)} e^{mu+nv} \qquad D \to m,\ D' \to n$$

$$\frac{1}{(m+n)(m+n-1)} e^{mu+nv}$$

$$= \frac{1}{(m+n)(m+n-1)} x^m y^n \quad PI.$$

e^{ax+by} Case I: $D \to a$, $D' \to b$

$e^{mu} \cdot e^{nv} = (e^u)^m (e^v)^n = x^m y^n$